Detection of Aliasing in Persistent Signals

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Abstract

In this paper we show that rather benign assumptions about an underlying signal permits us to detect aliasing in samples from that signal. This result which was inspired by work on stochastic signals, applies to single samples and does not require knowing something about samples from an ensemble of signals.

Keywords and phrases – Shannon sampling theorem, aliasing, a priori assumptions, sample statistics

AMS Mathematics Subject Classification – 94A20, 94A12

1 Introduction

Detection of aliasing from temporal samples alone, with no restrictions on the original continuous-time source, is impossible because any set of samples may be reconstructed (using convolution with the sinc function) to a properly sampled signal having the same samples. However, quite often additional information about the source is available. It is, of course, obvious that tight constraints on the source would permit perfect reconstructions of vastly under-sampled signals. For example, the constraint that the data comes from a linear function of time makes any two samples sufficient. A less extreme example is a signal with a lower as well as an upper frequency cutoff (a *bandpass signal*). For bandpass signals, it is well-known that one can sample at a rate below twice the highest frequency while still achieving perfect signal recovery (see [4, p.138, theorem 13.3]).

What are the weakest constraints that one can put on the signal and still get something—detection of aliasing, for example? Here, we examine constraints of stationarity. In 1988 Hinich and Wolinsky [3] suggested a bispectral test for detecting aliasing in temporally sampled stationary stochastic processes[1]. The test aroused some controversy [6, 1] which is examined in [2] and in [7]. In [7] we show, in detail, that the test *does* detect aliasing in some signal processes and that it is the constraint of stationarity that makes the detection of aliasing possible. Briefly, if we under-sample a stationary process and then reconstruct a continuous-time signal from the samples using the Shannon sinc filter, the reconstructed process will not, in general, be stationary. In contrast, a *proper* sampling followed by reconstruction will not destroy stationarity because this procedure just reconstructs the original signal. Detecting non-stationarity in the reconstructed process thus suffices to establish the existence of aliasing in the time series, provided it can be assumed that the original signal was stationary. These results are reviewed in Section 3.

Applying these concepts requires either a random sample of the paths of the process or an assumption of ergodicity which makes it possible to extract statistics from a single sample path. In this paper we attempt to generalize the results for stationary processes to the more common case where we have only a single sample path and can make no assumption of ergodicity. In other words, we look for ways to discover under-sampling in a time series drawn from a single waveform, which may or may not be a sample path of some underlying stochastic process. We define *sampling stationarity*, a form of stationarity that makes sense for single waveforms, and show that it can be used to detect aliasing in complex, continuous-spectrum signals. We present reasons to believe that sampling stationarity should be a generic[2] property

[1] In the following, we will use the terms *signal process* or just *process* for stochastic signal processes. Except when we use the terms *sample path* for a realization of a stochastic process or *random sample*, the word "sample" will refer to temporal sampling.

[2] We use the term *generic* in a nontechnical sense. The term usually occurs in a situation where one would like to say "with probability 1" but where no obvious probability measure exists.

of signals and that the destruction of sampling stationarity by the process of under-sampling and reconstruction should occur quite generally. Finally, we explain how it might be possible to use the reconstructed sample statistics plots (RSS plots) that we use to detect aliasing to obtain additional information about individual Fourier components beyond the Nyquist frequency.

The remainder of this paper proceeds as follows. After illustrating the key idea of this paper with an example in Section 2, we proceed, in Section 3, to demonstrate how a constraint of stationarity permits the detection of under-sampling in some signal processes. Then in Section 4 we define sampling stationarity. In Section 4.1 we use examples to show that the concept of sampling stationarity does, indeed, enable detection of aliasing for nontrivial signals. In Section 4.2 we consider the case of periodic signals. For this class of signals, we provide a complete explanation of how (and when) the method of high-frequency detection works. The possible extension of this explanation to non-periodic signals is then discussed in Section 4.3. In Section 4.4, we present some reasons to believe that the plots that we have used to detect aliasing may also be used to recover some portion of the original signal's high-frequency content. This is followed by suggestions for further work (Section 5) and a conclusion that summarizes the work in this paper (Section 6). Two after-notes contain computational and mathematical details.

2 Example

The key idea of our approach is captured by a very simple example. Suppose that we sample a square wave that takes the values -1 and 1. There is a unique properly band-limited signal that has this time series as its samples. We can compute this signal by applying the Shannon sinc filter to our time series. We may regard this computation as an attempt to reconstruct the original continuous-time signal. If we can reject this reconstructed signal as the source of our samples, then we must conclude that the time series contains aliased components.

Note that our given time series consists only of -1's and 1's. The reconstructed signal, on the other hand, is necessarily a continuous function of time, taking on all values in the interval $[-1, 1]$ (and, in fact, beyond). The only way that we could have obtained a sequence of -1's and 1's by sampling the reconstructed signal is if we had chosen a particular sampling rate (or one of its sub-harmonics) and a unique shift of the sampling comb. Any other combination of sampling rate and shift would have produced a series that takes a continuum of values. The probability of having chosen the special sample rate and shift that give a sequence of -1's and 1's is clearly zero, provided that our sampling rate was chosen independently of the source. With this proviso, then, we can reject (with confidence level 1) the hypothesis that our time series consisting of -1's and 1's came from sampling the reconstructed signal.

The assumption that the sampling rate was chosen independently of the source is justified in most (but not all) cases of practical importance because we can rule out any interdependence between the source and the sampling rate on physical grounds. For example, if a signal produced by a distant source is sampled at a predetermined rate, such a coupling is clearly out of the question—it would amount to believing that the process that produced the signal "knew" when we were going to sample at a distant location.

We can draw valid conclusions from a sampled signal about Fourier components beyond the Nyquist frequency only if we can put constraints on the original continuous-time source. How can we characterize the constraints that we are imposing in this case? Effectively, we are assuming that the sample times (which are determined by the sample rate and shift) do not play a distinguished role in the source. Showing that the sampling times are distinguished in the reconstructed signal then suffices to reject the reconstructed signal as the original source of the samples.

How, then, can we extend this analysis to more general classes of signals? In the case of a square wave (or any signal that takes on a finite number of values), the appearance of the time series produced by sampling the reconstructed signal at the given sampling times could not be more different from the appearance of a time series produced by sampling at any other shift of the sampling comb. Thus it is clear what we mean when we say that the sample times are distinguished in the reconstructed signal. For more general signals, however, it is not so clear exactly what it means for the sample times to be distinguished.

There is one obvious case in which we can be assured that the sample times are not distinguished in the original signal and in which we can detect the distinguished character of the sample times in the reconstructed signal. If the original signal is, in fact, a stationary signal process, then, by definition, *no* time is distinguished. The appearance of non-stationarity in the reconstructed signal would then indicate the presence of aliasing in the time series. The detection of aliasing in time series from stationary signal processes is the subject of the next section. Following that, we use our example of sampling from a square wave and insights from the case of stationary processes to develop a method for detection of aliasing in single waveforms.

3 Detection of Aliasing in Stationary Processes

Consider the case of detection of aliasing in stationary signal processes. We start with the simplest stationary processes imaginable—randomly shifted periodic signals. If we have a waveform, $x(t)$, with period T, then we can produce a stationary process by adding to t a random time shift, θ, that is evenly distributed on $[0, T)$. A sample path of our process then has the form $x(t+\theta)$ for a particular choice of θ.

Consider then the effect of under-sampling and reconstruction on a simple sine process,

$$x(t) = \sin(2\pi f t + 2\pi f \theta), \tag{1}$$

where θ is evenly distributed on $[0, f^{-1})$. If we under-sample with a sampling interval Δt, corresponding to the Nyquist band $[-(2\Delta t)^{-1}, (2\Delta t)^{-1})$, and then reconstruct via convolution with the sinc filter, we get the sine process given by

$$x_r(t) = \sin(2\pi \hat{f} t + 2\pi f \theta). \tag{2}$$

Here, $\hat{f}$ is the aliased frequency, given by $\hat{f} = f + k_f/\Delta t$ where k_f is the unique integer that places $\hat{f}$ in the Nyquist band. The key point is that the phase of the reconstructed signal is the same as the phase of the source even though the frequency has changed to the aliased value $\hat{f}$. For a process with a single harmonic, the reconstructed signal remains stationary because the phase term, $2\pi f\theta$, is evenly distributed on 2π.

Consider then a second signal process,

$$x(t) = \sin(2\pi\alpha t + 2\pi\alpha\theta) + \sin(2\pi\beta t + 2\pi\beta\theta), \tag{3}$$

where β is an integer multiple of α and θ is chosen randomly from the interval $[0, \alpha^{-1})$. Since the time shift, θ, is the same for both components, this is, for the various values of θ, just a shifted waveform of a given shape. Since θ is evenly distributed over the period, α^{-1}, the process is stationary.

If we sample this process at a rate low enough for both components to be aliased and then reconstruct using the sinc filter, we get

$$x_r(t) = \sin(2\pi\hat{\alpha} t + 2\pi\alpha\theta) + \sin(2\pi\hat{\beta} t + 2\pi\beta\theta), \tag{4}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the aliased frequencies. Although the phase terms, $2\pi\alpha\theta$ and $2\pi\beta\theta$, are still evenly distributed over 2π, they now correspond to *different* time shifts for the two components. Thus, we no longer have a single shifted waveform and we can expect, in general, that stationarity will have been lost.

We illustrate this loss of stationarity on an example by setting $\alpha = 1.0$ and $\beta = 3.0$ in Equation 3 and choosing a sample time, Δt, equal to $e = 2.71...$. We may detect the loss of stationarity by examining the envelope of the

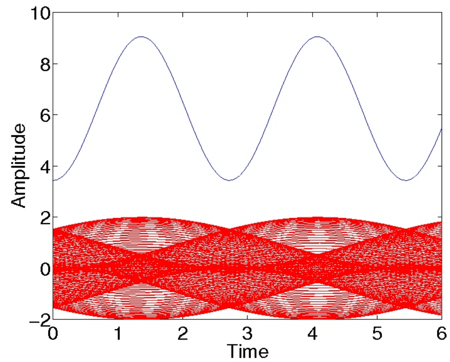

Figure 1: Plot illustrating the non-stationarity of a randomly shifted waveform that has been under-sampled and then reconstructed. The upper curve is the sixth moment[4]. The lower curve shows the process envelope. The original process is given by Equation 3 with $\alpha = 1.0$, $\beta = 3.0$, $\theta \in [0, 1)$, and $\Delta t = e = 2.71...$. (Another way of looking at the process envelope is that it is the area covered by graphs of *all* signals in the process – stationarity would imply no structure along the time axis).

reconstructed process. We define the envelope of a process, X_t, as the support of the probability density of X_t as a function of t. Another definition that would often coincide is that the process envelope at time t is the smallest interval containing the support of the process at time t. For a process produced by randomly shifting a periodic waveform, the envelope may be conveniently displayed by plotting the sample paths corresponding to a representative collection of time shifts as in Figure 1. Clearly, a stationary process must have a constant envelope. If we compute the envelope for the process defined in Equation 4 with the parameter values that we have specified, we get an oscillating figure (see Figure 1). This implies that the signal is

non-stationary. In fact, it is cyclostationary with period equal to the sampling interval.

As explained above, we do not lose stationarity when our original signal is a single sine wave. Nor do we lose stationarity when ratios between frequencies are preserved under the aliasing. For example, if $\Delta t = 1.0$ and the original frequencies are $(10/9, 20/9, 30/9)$, they would alias to $(1/9, 2/9, 3/9)$ and we would obtain another stationary process. But this situation is very special (non-generic). In general, more than one Fourier component is present and we do not have the special relationships between the sampling rate and the

[4] The sixth moment was chosen for clarity of presentation. The second moment remains constant in this case. The fourth moment does oscillate but the scale of its oscillation is too small to allow meaningful display of the moment and the envelope on the same scale.

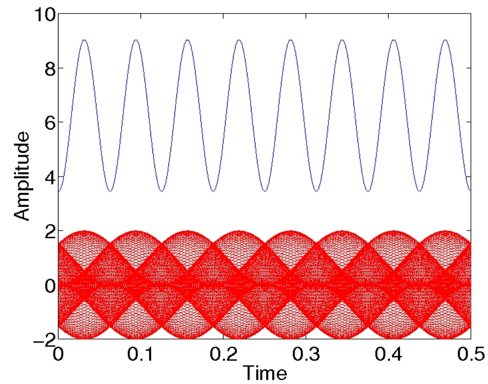

Figure 2: The sixth moment (upper curve) and the process envelope (lower curve) of the process given by Equation 3 with $\alpha = 0.25$, $\beta = 0.75$, $\theta \in [0, 4)$, and $\Delta t = 1.0$.

component frequencies that preserve ratios between frequencies when under-sampling. Thus, we expect that, generically, a stationary process formed by randomly shifting a periodic waveform will lose stationarity upon under-sampling and reconstruction.

Within the context of single shifted waveforms, the destruction of stationarity can occur in some remarkable situations. Consider that under-sampling and reconstruction can break stationarity even when only one of the components (β, say) is aliased and when β aliases to α or $-\alpha$. In other words, stationarity can be broken even when the two components, after under-sampling, lie right on top of each other. We can see this by choosing $\alpha = 0.25$, $\beta = 0.75$, and $\Delta t = 1.0$ in Equation 3. The envelope for the reconstructed process is shown in Figure 2 where the non-stationarity is apparent. (Of course, we cannot possibly detect this loss of stationarity by examining only a single sample path, since the sample path will never be more than a single sine wave of some amplitude and phase.)

Not all stationary processes are randomly shifted periodic waveforms. What can we say about more general stationary processes? It is clear that, if there exist *no* phase relationships between any of the Fourier components of the process, then under-sampling and reconstruction will not destroy stationarity. For a generic stationary process, though, we would expect at least some sets of components to exhibit phase relations. In that case, we would expect stationarity to be destroyed because it is difficult to imagine how the destruction of stationarity associated with one set of components could somehow be canceled out by the presence of other incommensurate components.

This argument, together with the observation that there is simply no reason to believe that stationarity should be preserved under under-sampling and reconstruction, suggests that the loss of stationarity should be a general feature of stationary processes.

4 Detection of Aliasing in Single Sample Paths

The method that we have just used to detect aliasing in a sampled stationary process requires complete knowledge of the discrete-time process obtained by sampling the original continuous-time source. Usually, however, we have available to us only a single sample path. Therefore, we require a method for detecting aliasing in a single waveform which may or may not be a sample path of a stochastic signal process.

We may develop such a method by reconsidering the example of sampling from a square wave discussed in Section 2 in light of our discussion of the effect of under-sampling on stationary signal processes. Recall that the sampled time series from the square wave takes on the values -1 and 1. We may state this in statistical language by saying that the one-time probability density of the time series consists of two Dirac delta functions centered at -1 and 1, respectively. Now, we would have obtained the same one-time statistics if we had sampled the original square wave with any shift of the sampling comb. We will say that a waveform has *sampling stationarity* for a given sampling interval if the one-time sample statistics do not change as the position of the sample comb is shifted along the waveform. *Observing that the original square wave had sampling stationarity for the given sampling interval is essentially equivalent to saying that the sample times were not distinguished in the source.*[5] Note that the signal obtained by applying the Shannon sinc filter to the time series does *not* have sampling stationarity—the one-time statistics of the reconstructed signal vary dramatically with shifts of the sampling comb (see Figure 3). This lack of sampling stationarity corresponds to the distinguished role of the sample times in the reconstructed signal. Of course, this distinguished role for the sample times is what allowed us to reject the reconstructed signal as a candidate for the original source of the samples and, thus, to conclude that the sampled series contained aliased components.

This discussion suggests the following test for aliasing in signals (no underlying stochastic process assumed). Collect the statistics on the recorded samples. Reconstruct the signal at various shifts of the sampling comb and collect the statistics at these reconstructed samples. Compare with the orig-

[5]Note that the original square wave does not have sampling stationarity for a sampling interval equal to its period. In general, a periodic signal will not have sampling stationarity with respect to sampling intervals commensurate with its period. However, the set of sampling intervals that are commensurate with a given period has Lebesgue measure zero. Clearly, the probability of choosing such a special sampling interval is 0 under the assumption that the sample times are chosen independently of the source.

inal statistics. (We use the term "statistics" loosely, without the assumption that the samples are independent samples of some underlying probability distribution.) If we find that the reconstruction has different statistics at some shift of the sampling comb, an assumption of sampling stationarity for the original signal implies that the reconstruction is not the original signal and therefore that the signal was under-sampled.

For this test to be at all useful, two questions must be answered:

1. Are typical signals characterized by sampling stationarity?

2. Do typical under-samplings reconstruct to signals for which sampling stationarity is violated?

The answer to question 1 is clearly "yes" for sample paths of ergodic stationary processes and for signals from ergodic dynamical systems. It is also clear that there are other classes of signals which possess sampling stationarity. For example, general periodic signals (not just square waves) possess sampling stationarity if the sampling interval is incommensurate with the signal period (a generic condition). Below, we conjecture that sampling stationarity is a generic property of signals.

The examples that we consider next suggest that the answer to the second question is also "yes".

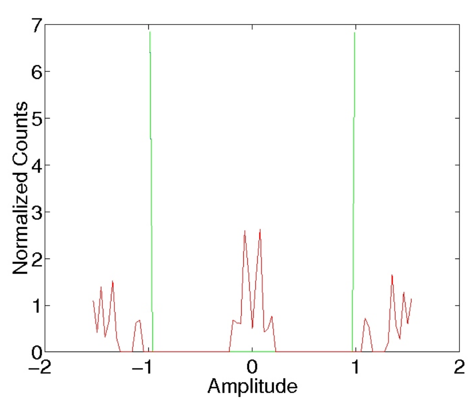

Figure 3: Sample statistics of data and reconstruction from a square wave. The plot shows sample statistics of the data in blue and green (which are indistinguishable) and the reconstruction in red. (See the beginning of Section 4.1 for an explanation of the blue and green histograms.) The red histogram is obviously very different from the blue and green with which it would coincide if the reconstruction had sampling stationarity. The blue and green histograms have been rescaled so as to make the three histograms of comparable height.

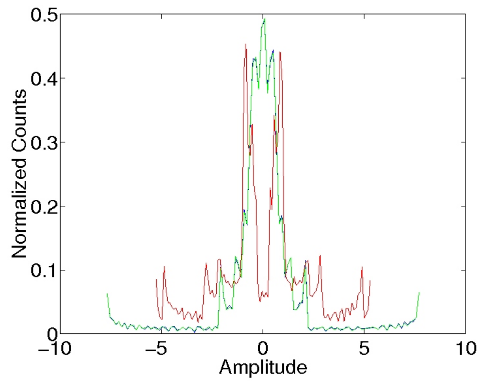

Figure 4: Sample statistics of data and reconstruction from a periodic signal. The blue and green histograms coincide, indicating that the original signal had sampling stationarity. The red histogram, showing the statistics of the reconstructed signal, is obviously very different from the blue, showing that the reconstruction does not have sampling stationarity.

4.1 Examples

In each of the examples listed below, the time series to which we apply our test for aliasing was split into two interleaving series, D_1 from the samples taken at [0,2Δt,4Δt, ...] and D_2 from the samples taken at $[\Delta t, 3\Delta t, 5\Delta t, ...]$. The sample statistics corresponding to D_1 and D_2 are plotted in blue and green, respectively. For original signals with sampling stationarity, these two histograms will coincide. We then produce a reconstruction from D_1, computed at the times corresponding to D_2. The sample statistics corresponding to this reconstructed series are shown in red. If the red histogram is significantly different from the blue, then the reconstructed signal does not have sampling stationarity.

Example: For a periodic signal, the generic condition of incommensurability of the sampling interval and the signal period implies that the signal has the property of sampling stationarity. But we also find that under-sampling and reconstructing produces a signal that does NOT have sampling stationarity, as illustrated in Figure 4. The data for the plot were generated by sampling ($\Delta t = e/16$) a sum of sines with frequencies $(0, 1, 2, ..., 10)$ and random amplitudes that ranged between .78 and 1.22.

Example: If our signal consists of a sum of sine waves with incommensurate frequencies, then we cannot detect aliasing by this method (see Figure 5). Although such a signal will have sampling stationarity, the sampling stationarity will not be broken by under-sampling and reconstruction because it is impossible to have relationships between the phases of different Fourier com-

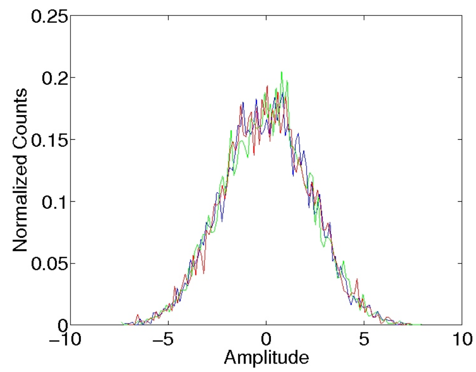

Figure 5: Sample statistics of data and reconstruction from a sum of sine waves with incommensurate frequencies. The blue and green histograms coincide. The red, showing the statistics of the reconstructed signal, is NOT obviously different.

ponents (see Section 4.2).

Example: The previous example might lead to the suspicion that this method works only for periodic signals (or step signals such as the square wave). However, the presence of incommensurate Fourier components does not necessarily destroy the ability to detect aliasing in a periodic waveform with more than one harmonic component. Figure 6 shows the result of combining a periodic waveform with incommensurate harmonics. The total power in the incommensurate harmonics is about 21% of the power in the periodic waveform. The sampling stationarity of the original signal and the breakdown of sampling stationarity with under-sampling and reconstruction are apparent. This shows that, as long as *some* of our aliased Fourier components are commensurate with other components, the method can work.

Example: So far, we have demonstrated that the method works for pure periodic signals and for periodic signals mixed with incommensurate harmonics. Figures 7, 8, and 9 show that the method works for much more complex signals with continuous spectra. The signals are taken from the Lorenz and Rössler systems (see section A).

The success in detecting aliasing in time series from the Lorenz and Rössler systems suggests that the method may work for a very broad class of signals. Before attempting to determine how wide this class might actually be, we will look at the periodic case in order to begin to understand the precise mechanism of the method.

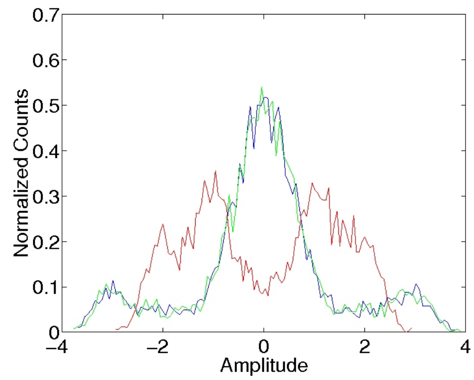

Figure 6: Sample statistics of data and reconstruction from a mixture of a periodic waveform and incommensurate harmonics. The blue and green histograms coincide. The red, showing the statistics of the reconstructed signal, is obviously different.

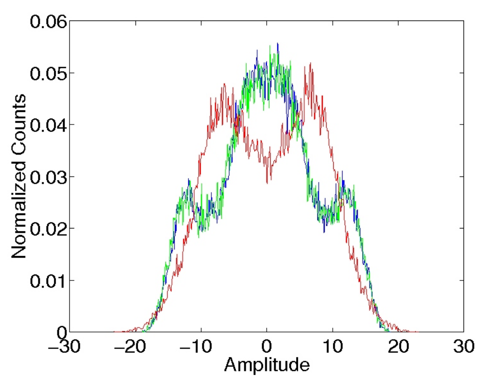

Figure 7: Sample statistics of data and reconstruction from the x-coordinate of the Lorenz model. The blue and green histograms coincide and the red is clearly different.

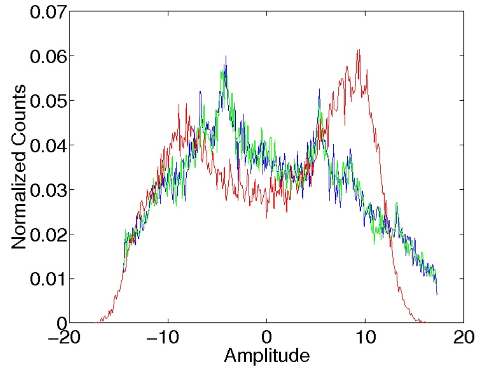

Figure 8: Sample statistics of data and reconstruction from the x-coordinate of the Rössler model. The blue and green histograms coincide and the red is clearly different.

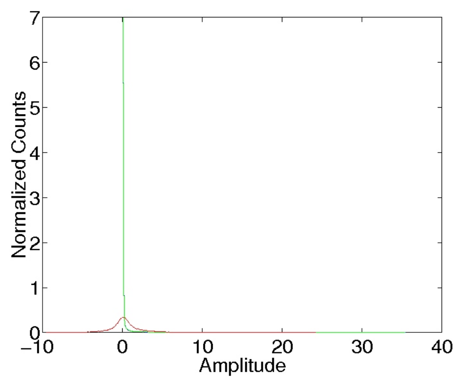

Figure 9: Sample statistics of data and reconstruction from the z-coordinate of the Rössler model. The blue and green histograms coincide and the red is clearly different.

4.2 A Closer Look at the Periodic Case

If one samples a periodic signal incommensurately with the signal period, the samples end up mixing evenly around the waveform (see section B). Thus, all shifts of a sampling comb with a sampling interval that is incommensurate with the period will produce the same statistics. This implies that:

Theorem 4.1. *A periodic signal will have sampling stationarity with respect to any sampling interval that is incommensurate with the period of the signal.*

Conversely, sampling with an interval that *is* commensurate with the period will, in general, produce statistics that depend on the sampling shift. Theorem 4.1 implies that:

Theorem 4.2. *Every periodic signal has sampling stationarity with respect to all sampling intervals except for a set of intervals with (Lebesgue) measure zero.*

Thus, the probability of choosing a sampling interval for which a given periodic signal does not have sampling stationarity is zero, provided the interval is chosen independently of the signal.

What, then, is the effect of under-sampling and reconstruction on this sampling stationarity? The sample statistics are determined by the shape of the waveform (see section B for the exact formula). It can be shown that the process of under-sampling and reconstruction is equivalent to sampling the original waveform at the same rate with the individual Fourier components shifted with respect to each other. When different components experience different time shifts, the shape of the effective waveform changes. Consequently, the statistics change. We now explain this in detail.

When we sample a single harmonic with frequency f and phase φ every Δt time units, we get the values

$$y_n = \sin\left(2\pi f n \Delta t + \varphi\right) \qquad n \in Z. \tag{5}$$

We will temporarily suppress the phase and rewrite this expression as

$$\begin{aligned} \sin\left(2\pi f n \Delta t\right) &= \sin\left(2\pi f n \Delta t + 2\pi k n\right) \\ &= \sin\left(2\pi\left(f + \frac{k}{\Delta t}\right) n \Delta t\right) \end{aligned} \tag{6}$$

for any integer k, so that the reconstruction of this component at points $1 + s, 2 + s, 3 + s, \ldots$ is given by

$$\hat{y}_{n,s} = \sin\left(2\pi\left(f + \frac{k_f}{\Delta t}\right)(n + s)\,\Delta t\right), \tag{7}$$

where the reconstruction chooses precisely one of the integral k's, which we will call k_f, such that $f + k_f/\Delta t$ is in the interval $[-1/2\Delta t, 1/2\Delta t)$. We can now rewrite the reconstructed harmonic as

$$\begin{aligned}
\hat{y}_{n,s} &= \sin\left(2\pi\left(f + \frac{k_f}{\Delta t}\right) n\Delta t + 2\pi\left(f + \frac{k_f}{\Delta t}\right) s\Delta t\right) \\
&= \sin\left(2\pi f n\Delta t + 2\pi k_f n + 2\pi f s\Delta t + 2\pi k_f s\right) \\
&= \sin\left(2\pi f n\Delta t + 2\pi f s\Delta t + 2\pi k_f s\right)
\end{aligned} \tag{8}$$

where we drop $2\pi k_f n$ since k_f is an integer. *Thus, the reconstructed signal has samples at a shift, s, as though we were sampling the original waveform, but with the phase of the individual Fourier component shifted by the amount $2\pi f s\Delta t + 2\pi k_f s$.* The first term amounts to a time shift which is the same for all the components in the waveform. This implies that these first terms do not change the shape of the waveform and can be ignored. So we may consider the effective waveform (at a shift s) to be

$$\sum_i A_i \sin(2\pi f_i n\Delta t + 2\pi k_{f_i} s + \varphi_i) \tag{9}$$

where we have reinserted the phase. The term $2\pi k_{f_i} s$ amounts to a time shift that is different for different f_i. This difference in time shifts leads to a change in the shape of the effective waveform as s changes which in turn changes the sample statistics.

For a given waveform, it is clear that almost any change in the shape of the waveform will change the sample statistics. (For example, a generic choice of s will change the heights of the extrema, changing the locations of the singularities in the histogram. See section B.) Thus, we conclude that *generically, periodic signals have sampling stationarity which is destroyed by under-sampling and reconstruction.*

4.3 Non-periodic Signals

Now, we want to use the insight that we have gained for the case of periodic signals to get a better understanding of the answers to the two questions at the end of Section 4. We begin with some general questions about the kinds of signals to which our method might possibly apply.

Consider first the case of transient signals. In order to be able to talk about sampling stationarity at all, we have to be able to take as many samples as we want (at the given sampling rate) in order to be able to estimate the one-time probability distribution to arbitrary accuracy. This implies that we must think of our signals as functions of infinite time. In this context, any transient signal has trivial sampling stationarity—the probability distribution is a delta function at zero. By the same token, under-sampling and reconstruction will not destroy this sampling stationarity. Thus, we need to restrict our attention to persistent (non-transient) signals.

Within the class of persistent signals, it is clear that we need the signals that we consider to have well-defined sample statistics for arbitrary sampling intervals and shifts. Given that we are discussing aliasing, our signals also need to have a Fourier transform (in some sense). The set of signals with well-defined power spectra (which will have, in general, singular components) will clearly meet these criteria, although the actual class to which our method applies may be larger. In the following, then, we may take the term *persistent signal* to refer to a signal with a well-defined, nonzero power spectrum.

Consider then the question of which signals have sampling stationarity for which sampling intervals. In the case of periodic signals, Theorems 4.1 and 4.2 provide what is essentially a complete answer—sampling stationarity holds for generic choices of signals and sampling intervals. At first glance one might try to generalize Theorem 4.1 to the following:

Conjecture 4.1. *Every signal has the property of sampling stationarity for every sampling interval Δt that is not commensurate with the period of any* singular *component of its spectrum.* (FALSE)

Unfortunately this conjecture is false as may be seen from the following counterexample. If we under-sample and reconstruct a signal with a purely continuous spectrum (such as our signal from the Lorenz system), we will introduce no new singular components. Thus, the reconstructed signal will have a purely continuous spectrum. If the conjecture were true, then, such a reconstructed signal would have sampling stationarity for all sampling intervals by virtue of having no singular components. Yet it is just the *lack* of sampling stationarity of this signal with respect to the given sampling interval that allows us to detect aliasing in this case. Thus, we know that there exist signals that lack sampling stationarity with respect to sampling intervals that are not commensurate with any singular component of their spectra and the conjecture is false. However, the reconstruction of an under-sampled signal has a very special relationship to the interval with which the sampling was done. Thus, one expects that re-sampling the reconstruction with a new sampling interval *not related to the original interval* will yield statistics that are again stationary with respect to shifts in the sampling comb. Therefore, we arrive at the following conjecture:

Conjecture 4.2. *Every signal has the property of sampling stationarity for* every *sampling interval Δt,* except *a set of Δt's with (Lebesgue) measure zero.*

This conjecture implies that, if one were to observe the reconstructed statistics varying with changes in the shift, this observation would be enough to conclude (with probability 1) that the samples came from an under-sampled waveform. In other words, *the truth of the conjecture would imply that the detection of under-sampling by the proposed method is generically free of false positives*

Next, we want to know when under-sampling and reconstruction of persistent non-periodic signals will yield new signals which *have* the property of sampling stationarity. (In other words, we also want to know when we can get false negatives.) The analysis that we have presented for periodic signals *suggests* that under-sampling and reconstruction should destroy sampling stationarity for general persistent signals in which at least some of the aliased Fourier components are commensurate with other components of the signal[6]. The reasoning is that each individual component can be regarded as a part of a family of harmonics and that the effective shape of the waveform associated with this family is changing with shifts of the sampling comb. There does not appear to be any reason to believe that combining different periodic waveforms, each of which is changing its sample statistics with shifts of the sampling comb, would result in sample statistics that do not change. Therefore, *it seems likely that, generically, persistent signals have sampling stationarity that is destroyed by under-sampling and reconstruction.* In order to turn this last statement into a well defined conjecture, it will be necessary to define precisely what is meant by "generically" in the case of persistent non-periodic signals. The question of exactly how to define "persistent" must also be answered. Since the transformation that takes us from a waveform to sample statistics is extremely nonlinear, a proof is likely to be difficult.

4.4 Recovery of High-Frequency Information

The next question that presents itself is whether or not we can recover information about individual aliased Fourier components using the sampling-shift dependence of the reconstructed statistics. Ideally, we would like to know how much of the signal at an individual frequency, f, in the Nyquist band comes from each frequency that aliases to f.

Consider the one-time probability density of the reconstructed signal, $p(x)$, as a function of both x and shift s. We call this two-dimensional surface a Reconstructed Sample Statistics (RSS) plot (see figures 10 and 11 for example RSS plots).

The RSS plot has dependencies on s tied directly to the quantities k_{f_i}. Each k_{f_i}, in turn, determines the particular copy of the Nyquist band in which its corresponding f_i is located. This chain of dependencies suggests that the

[6]Note that the condition that the original signal must have harmonically related components (i.e. commensurate components) will be satisfied by any signal with a nonzero continuous part to its spectrum as well as by periodic signals.

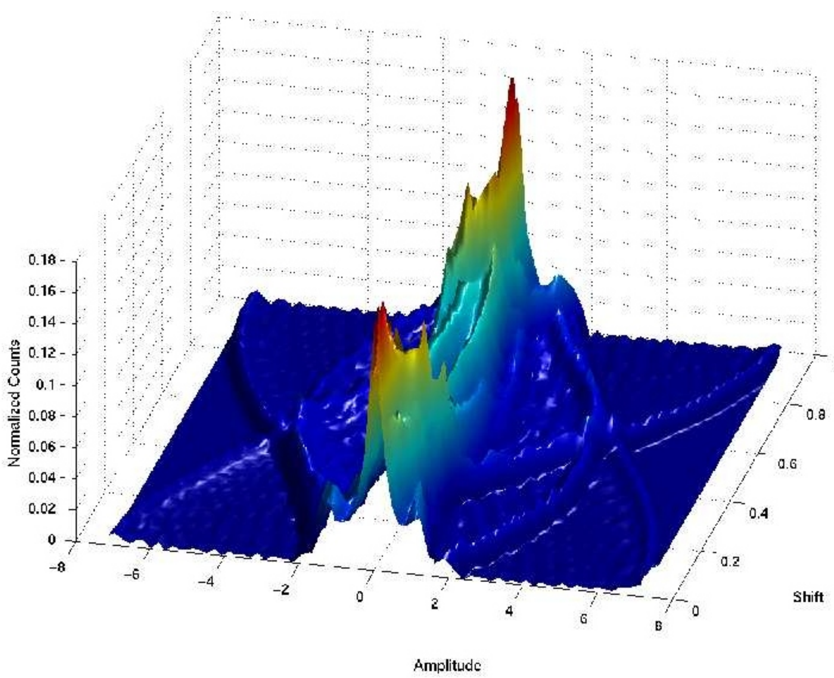

Figure 10: Example RSS Plot: A signal $f(t) = \sum_{i=0}^{10} c_i \sin(2\pi i t)$ with the c_i's chosen randomly in $[.75, 1.25]$, was sampled every $.0625 * \exp(1)$ time units ($=$ .1698... time units). Since the Nyquist $\Delta\tau$ is is .05 time units, the signal is *undersampled* and the as the plot shows, the sample statistics change with the reconstruction shift.

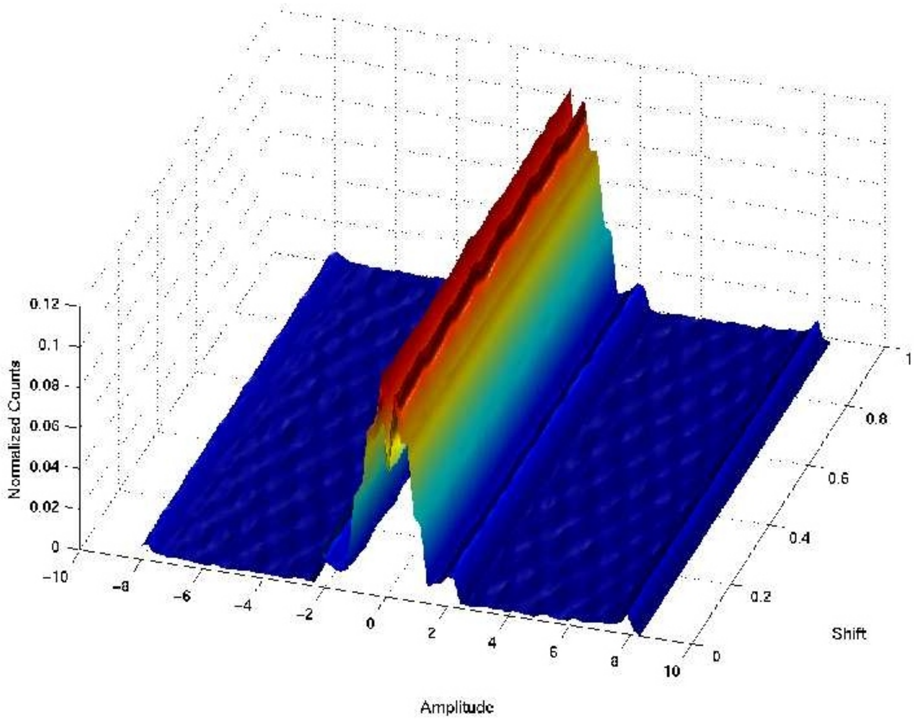

Figure 11: Another Example RSS Plot: A signal $f(t) = \sum_{i=0}^{10} c_i \sin(2\pi i t)$ with the c_i's chosen randomly in $[.75, 1.25]$, was sampled every $.0125 * \exp(1)$ time units (= .03398... time units). Since the Nyquist $\Delta\tau$ is is .05 time units, the signal is *properly sampled* and as the plot shows, the sample statistics **do not** change with the reconstruction shift.

RSS plot contains the information necessary to determine the contribution of each band to the signal at a given frequency in the Nyquist band. The inverse problem is greatly complicated by the interaction of the Fourier components and the nonlinear "projection" that turns the waveform into statistics. This nonlinear inverse problem will be a major focus of future work on the detection (and possibly correction) of aliasing.

5 Directions for Further Investigations

In addition to the work already alluded to on the inverse problem formed by the RSS plots, there are other issues to explore. Included among them are:

- What are the effects of noise on this method for detection of aliasing?
- What is the effect of near commensurability of sample interval and signal period?
- What is the effect of finite time-series length?
- How does the departure of the statistics of the reconstructed signal from stationarity depend on the fraction of the total power that lies outside the Nyquist band?

These questions are important to the practical usefulness of the method of high-frequency detection/recovery.

6 Conclusion

Although the idea of detection of aliasing is typically dismissed with references to the Nyquist criterion and the Shannon reconstruction theorem, we have demonstrated that detection of aliasing is possible with what appear, at first glance, to be very weak prior assumptions. The key concept is that of sampling stationarity. We emphasize that *this concept makes sense for single waveforms.* Although this concept arose in the consideration of step signals like square waves, its usefulness extends far beyond these signals. In particular, our method enables the detection of aliasing in samples from nontrivial waveforms such as measurements from motion on the Lorenz or Rössler attractors. As indicated above, many questions remain. Some of these are important for the practical utility of the concept of sampling stationarity and the associated RSS plots.

7 After-notes: Computational Details

The calculations represented in the paper were done with Matlab. The Lorenz equations,

$$\begin{aligned}\dot{x} =&\sigma(y-x)\\ \dot{y} =&x(R-z)-y\\ \dot{z} =&xy-bz,\end{aligned}\tag{10}$$

were integrated with parameter values of $\sigma = 10$, $R = 28$, and $b = 8/3$ using Matlab's "ODE45" which is an adaptive step size routine. Relative tolerance was set to the default value of 1.0×10^{-3} and absolute tolerance was set to the default 1.0×10^{-6}. Initial conditions were set at $x = y = z = 1$. Values for the x, y, and z coordinates were saved every 0.5 time units. 200,001 samples were taken and split into two interleaving time series each 100,000 samples long. The first series was used to reconstruct a signal via convolution with a sinc filter of length 200,001. These very long series and filters were used to minimize the effects of truncating the convolution at the ends of the series. The histograms for the reconstructed signal were computed from the middle 50% of the reconstructed series.

The Rössler equations,

$$\begin{aligned}\dot{x} =&-z-y\\ \dot{y} =&x+ay\\ \dot{z} =&b+z(x-c),\end{aligned}\tag{11}$$

were integrated in the same way with a sampling interval of 10 time units, and parameter values of $a = 0.15$, $b = 0.2$, and $c = 10$. In this way 200,001 samples were obtained and the splitting and reconstruction were done as described for the Lorenz equations.

All the histograms presented here were originally calculated from much shorter time series ($10,000$ samples). The features that allow us to conclude that aliased components are present were all clearly visible in the histograms made from shorter time series although, of course, the histograms were considerably rougher. We conclude that the results that we have presented are certainly not an artifact of finite-length time series.

8 After-notes: Sample Statistics for a Periodic Signal

If one samples a periodic signal, $h(t)$, incommensurately with the signal period T, the samples end up mixing evenly around the waveform[7]. The resulting histogram is proportional to the reciprocal of the derivative of the waveform. This follows from the fact that the probability of getting any particular t (position along the waveform) is uniformly distributed over $[0, T)$ which in turn implies that the probability of the interval $[y, y + dy)$ is the probability of the corresponding dt or $(1/T)(dy/h'(t))$. More precisely, the probability density for y is

$$p(y) = \frac{1}{T} \sum_{t_\alpha \in \mathcal{T}(y)} (h'(t_\alpha))^{-1} \tag{12}$$

where

$$\mathcal{T}(y) = \{t \mid t \in [0, T), h(t) = y\} \tag{13}$$

Note that the density will have $1/\sqrt{(y)}$ singularities at the local maxima and minima of $h(t)$. The form of the singularities follows from the fact that a generic waveform has maxima and minima with nonzero second derivative.

References

[1] G. Frazer, A. Reilly, and B. Boashash. The bispectral aliasing test. In *Proceedings IEEE Workshop on Higher-Order Statistics*, pages 332–335, June 1993.

[2] Melvin J. Hinich and Hagit Messer. On the principal domain of the discrete bispectrum of a stationary signal. *IEEE Transactions on Signal Processing*, 43(9):2130–2134, 1995.

[3] Melvin J. Hinich and Murray A. Wolinsky. A test for aliasing using bispectral analysis. *Journal of the American Statistical Association*, 83:499–502, June 1988.

[4] J.R.Higgins. *Sampling Theory in Fourier and Signal Analysis*. Oxford University Press, 1996.

[7] Proving this is surprisingly difficult. An equivalent problem is the determination of the density of points obtained by repeated iterations of an irrational rotation on a circle of unit circumference. The resulting distribution of points satisfies $n_{[a,b)}/N \approx (b - a)$ where $n_{[a,b)}$ is the number of iterates in $[a, b)$, N is the total number of iterates, and $[a, b) \subset [0, 1)$. See [5, p.39-40,29] for details.

[5] Ya. G. Sinai. *Introduction to Ergodic Theory*. Mathematical Notes. Princeton University Press, 1976.

[6] A. Swami. Pitfalls in polyspectra. In *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing*, volume IV, pages 97–100, 1993.

[7] Kevin R. Vixie, Murray Wolinsky, and David Sigeti. The bispectral aliasing test: A clarification and some key examples. In M. Deriche, B. Boashash, and W. W. Boles, editors, *Proceedings of the Fifth International Symposium on Signal Processing and its Applications (ISSPA'99)*, Brisbane, Australia, 1999. Signal Processing Research Centre, Queensland University of Technology. http://xxx.lanl.gov/abs/chao-dyn/9906020.

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